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# Modulational instability and exact solutions of the modified quintic complex Ginzburg–Landau equation

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**Abstract**

We have formulated a model of an equation that governs dynamics of nonlinear waves in many non-equilibrium systems. Based on this new equation, we have reviewed the well-known Lange and Newell's criterion for modulational instability of Stoke waves. Some exact solutions of this wave equation have also been found through a proper combination of the Painlevé analysis and Hirota's bilinear technique.

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## 1. Introduction

Recent developments in mathematical physics enable us to find solutions of nonlinear equations which appear in various fields of physics. Those solutions play important roles in understanding the fundamental properties of physical systems [1]. Thus several mathematical models have been proposed for the theoretical treatment of nonlinear wave propagation in one, two- and three-dimensional media. For instance, we consider physical systems described by the quintic complex Ginzburg–Landau equation (QCGLE) [2, 3]

$$\mathrm{i}U_t + PU_{xx} = Q_1|U|^2U + Q_2|U|^4U + \mathrm{i}\gamma U \tag{1}$$

in which $U(x,t)$ represents the complex wave amplitude of the phenomenon under examination, subscripts $t$ and $x$ stand for partial derivatives and $\mathrm{i}^2 = -1$. The QCGLE is of interest in many branches of physics. It is a one-dimensional model for large-scale behaviour of many non-equilibrium pattern-forming systems [4]. Relation (1) is also used as an envelope equation for describing a weakly subcritical bifurcation to counter-propagating

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waves [5]. Moreover, the QCGLE also accounts for the slow modulations of an oscillatory mode close to a subcritical bifurcation when $Q_{1i} > 0$, $Q_{2i} < 0$ and $\gamma = 0$ [6]. Although this equation has been the subject of intensive numerical investigations [7], it should be stressed that only a few works seeking its analytical solutions have been done [3].

We shall mention that the exact form of the QCGLE that represents a given physical process is obtained by a balance of various contributing factors under specified conditions. Here we thought that it would be of great interest, at least for the sake of comparison, to generalize equation (1) in order to allow it to describe physical systems in which many of the contributing factors are simultaneously operating. Owing to these, we introduce in this paper a modified form of the QCGLE, i.e.,

$$\mathrm{i}U_t + PU_{xx} = Q_1|U|^2U + Q_2|U|^4U + C(U_xU_x^*/U^*) + \mathrm{i}\gamma U. \tag{2}$$

This equation is henceforth called the modified quintic complex Ginzburg–Landau equation (MQCGLE). In equation (2), parameters $P$, $Q_1$, $Q_2$, $C$ and $\gamma$ are constants that shall be treated as complex quantities for the sake of generality. Then many types of dispersive and dissipative effects are included in this equation.

The MQCGLE is rather general as it includes dispersive and nonlinear effects in both conservative and dissipative forms. Indeed, it contains the well-known complex Ginzburg–Landau equation (CGLE) (case where $Q_2, C = 0$ and $\gamma$ is real) which possesses a large range of applications describing phase transitions and wave propagation in many non-equilibrium systems [8]. When $Q_2 = 0$, equation (2) becomes the modified CGLE that governs wave propagation and describes the development of unstable waves in non-equilibrium systems [9]. Equation (2) can be used to study the Benjamin–Feir turbulence in convective binary fluid mixtures near the critical point since it contains at least the quintic nonlinear term whose existence was predicted by Brand *et al* [10], the $C$ factor term will reveal the contribution of other unknown phenomena which deal with the system.

It is important to note that equation (2) differs from the previously existing forms of the QCGLE mainly by the $C$ factor term which was absent in the earlier works. Let us mention that this term, which is mathematically nonlinear, becomes under the geometrical-optics ansatz $\partial/\partial x \rightarrow \mathrm{i}k$ and $\partial/\partial t \rightarrow -\mathrm{i}\omega$ a linear term in the resulting dispersion relation. Thus despite its nonlinear form, this term reduces the nonlinear effects reinforced in the system by the presence of the quintic term.

This paper is outlined as follows. In section 2, the modulational instability (stability) is investigated. A new criterion is expected. In section 3, we look for some exact solutions of the MQCGLE. The last section is devoted to discussions and concluding remarks.

## 2. Modulational instability

Many nonlinear systems exhibit an instability that leads to the self-induced modulation of an input plane wave with the subsequent generation of localized pulses [11]. Well known as modulational instability, this phenomenon is responsible for various physical interesting effects such as the formation of envelope solitons in electrical transmission lines [12, 13], nonlinear optical fibres [14], dielectric media [15] and cavitons in plasma [16] as well as the filamentation of laser beams [17] and the break-up of monochromatic ocean waves [18]. More recently, the first experimental observation of the modulational instability was reported by Tai *et al* [19] using single-mode fibres. They showed how this phenomenon can be exploited to generate a soliton train at high repetition rate.

Thus, it becomes of interest to find under which conditions isolated pulses could be formed during the evolution of the wave in the system. So, to analyse the modulational instability in the framework of the MQCGLE, we consider a first-order perturbation of an harmonic wave [20, 21] and research the conditions of stability (instability). The MQCGLE admits travelling wave solutions

$$U(x,t) = U_0 \exp[\mathrm{i}(kx - \omega t)] \tag{3}$$

provided the nonlinear dispersion relation

$$\omega = \omega(k, |U_0|^2) = (P + C)k^2 + Q_1|U_0|^2 + Q_2|U_0|^4 + \mathrm{i}\gamma \tag{4}$$

is satisfied. We refer to $U_0$, $k$ and $\omega$, respectively as the complex amplitude, the wavenumber and the angular frequency of (3). The linear stability of the nonlinear plane wave can be examined by looking for solutions in the form [21, 22]

$$U(x,t) = [1 + B(x,t)]U_0 \exp[\mathrm{i}(kx - \omega t)] \tag{5}$$

where the complex quantity $B(x,t)$ is assumed to be small in comparison with the amplitude of the carrier. After substitution of (5) into equation (2), we linearize the result with respect to $B(x,t)$ and obtain the evolution equation for the perturbation

$$\mathrm{i}B_\tau + P(B_{xx} + 2\mathrm{i}kB_x) = Q_1(B + B^*)|U_0|^2 + 2Q_2(B + B^*)|U_0|^4 - \mathrm{i}kCB_x \tag{6}$$

in which $B^*$ is the complex conjugate of $B$. Solution of equation (6) can be taken as

$$B(x,t) = B_1 \exp[\mathrm{i}(lx + \Omega t)] + B_2^* \exp[-\mathrm{i}(lx + \Omega^* t)]. \tag{7}$$

In relation (7), $B_1$ and $B_2$ are complex constant amplitudes; $\Omega$ and $l$ represent respectively the angular frequency and the wavenumber of the perturbation. Insertion of (7) into equation (6) results in a linear homogeneous system for $B_1$ and $B_2$:

$$\begin{cases} (\Omega + y + r)B_1 + yB_2 = 0 \\ zB_1 + (\Omega + s + z)B_2 = 0 \end{cases} \tag{8}$$

with

$$\begin{aligned} y &= Q_1|U_0|^2 + 2Q_2|U_0|^4 \qquad & r &= P(l^2 + 2kl) + klC \\ z &= -Q_1^*|U_0|^2 - 2Q_2^*|U_0|^4 \qquad & s &= P^*(-l^2 + 2kl) + klC^*. \end{aligned} \tag{9}$$

System (8) has non-trivial solutions if the frequency $\Omega$ obeys the relation

$$\Omega^2 + b\Omega + e = 0 \tag{10}$$

wherein $b = s + r + y + z$ and $e = rs + rz + ys$.

At this level, expression (4) is utilized to better write the discriminant (delta) of (10) which takes the form

$$\Delta = X_1 + \mathrm{i}Y_1 \tag{11}$$

with

$$\begin{aligned} X_1 = {} & 2l^2(P_rQ_{1r} + P_iQ_{1i})|U_0|^2 + 4l^2(P_rQ_{2r} + P_iQ_{2i})|U_0|^4 + l^4P_r^2 - k^2l^2C_i^2 \\ & - 4Q_{2i}^2|U_0|^8 - 2l^2k^2P_i^2 - Q_{1i}^2|U_0|^4 + 4k^2l^2\left(P_i^2 + C_i^2\right) - 4k^2l^2(P_i + C_i)^2 \\ Y_1 = {} & 4kl^3P_rP_i + 8lkP_iQ_{2r}|U_0|^4 + 4lkP_iQ_{1r}|U_0|^2 + 2kl^3P_rC_i \\ & + 2lkC_iQ_{1r}|U_0|^2 + 4lkC_iQ_{2r}|U_0|^4. \end{aligned}$$

Solutions of equation (10) are the following complex parameters:

$$\Omega_1 = \alpha + \mathrm{i}\beta + (X_1 + \mathrm{i}Y_1)^{1/2} \qquad \text{and} \qquad \Omega_2 = \alpha + \mathrm{i}\beta - (X_1 + \mathrm{i}Y_1)^{1/2} \tag{12}$$

in which

$$\alpha = -kl(C_r + 2P_r) \qquad \text{and} \qquad \beta = (k^2 - l^2)P_i + k^2C_i + \gamma_r - Q_{2i}|U_0|^4. \tag{13}$$

Here, two cases are possible related each to the sign of $Y_1$. So when $Y_1 < 0$, the roots of $\Delta$ (i.e., $h_1$ and $h_2$) help to write explicitly frequencies $\Omega_1$ and $\Omega_2$

$$\Omega_1 = \alpha + \mathrm{i}\beta + h_1 - \mathrm{i}h_2 = (\alpha + h_1) + \mathrm{i}(\beta - h_2) \tag{14}$$

$$\Omega_2 = \alpha + \mathrm{i}\beta - h_1 + \mathrm{i}h_2 = (\alpha - h_1) + \mathrm{i}(\beta + h_2) \tag{15}$$

with

$$h_1 = \sqrt{\tfrac{1}{2}\left(X_1 + \sqrt{X_1^2 + Y_1^2}\right)} \qquad \text{and} \qquad h_2 = \sqrt{\tfrac{1}{2}\left(-X_1 + \sqrt{X_1^2 + Y_1^2}\right)}.$$

In the case where $Y_1 > 0$, calculations yield the frequencies

$$\begin{aligned} \tilde{\Omega}_1 &= \alpha + \mathrm{i}\beta - h_1 - \mathrm{i}h_2 = (\alpha - h_1) + \mathrm{i}(\beta - h_2) \\ \tilde{\Omega}_2 &= \alpha + \mathrm{i}\beta + h_1 + \mathrm{i}h_2 = (\alpha + h_1) + \mathrm{i}(\beta + h_2) \end{aligned} \tag{16}$$

and lead to solutions which have the same asymptotic behaviour as those derived from (14) and (15).

Since the frequencies $\Omega_1$ and $\Omega_2$ are complex quantities, it is not obvious how to specify their sign. But their imaginary parts contribute to increase the effects of perturbation in the system. Substitution of (14) into relation (7) permits us to understand the behaviour of $B(x, t)$. Because $h_2$ is always positive, $\beta - h_2 < \beta + h_2$ holds and then this behaviour is the function of the sign of $\beta - h_2$ which represents the imaginary part of $\Omega_1$. Indeed, we have

$$B_1\,\mathrm{e}^{\mathrm{i}\Omega t} = B_1\,\mathrm{e}^{\mathrm{i}\Omega_1 t} = B_1\,\mathrm{e}^{\mathrm{i}(\alpha+h_1)t}\,\mathrm{e}^{-(\beta-h_2)t} = B_1\,\mathrm{e}^{\mathrm{i}(\alpha+h_1)t}\,\mathrm{e}^{(h_2-\beta)t}.$$

Hence, it becomes clear that the asymptotic behaviour of (7) is related to the sign of the constant $\beta - h_2$.

If $\beta < 0$, then the quantity $h_2 - \beta$ is always greater than zero and solution (7) increases exponentially when $t$ tends to infinity. The system remains unstable under the modulation. But if $\beta > 0$, the behaviour of (7) will depend on the sign of $h_2 - \beta$. Two cases appear:

$$\text{when} \quad h_2 - \beta > 0 \qquad \text{i.e.} \qquad \mathrm{Im}(\Omega_1) < 0 \tag{17}$$

solution (7) diverges without limit as $t$ increases and the system is said to be modulationally unstable. Therefore, the difference $\beta - h_2$ can be written as

$$\begin{aligned} \beta - h_2 = \beta - \Big\{\tfrac{1}{2}\big[&-2l^2(P_r Q_{1r} + P_i Q_{1i})|U_0|^2 - 4l^2(P_r Q_{2r} + P_i Q_{2i})|U_0|^4 \\ &- l^4 P_r^2 - 4k^2 l^2\left(P_i^2 + C_i^2\right) + X_2\big]\Big\}^{1/2} \end{aligned}$$

with

$$X_2 = k^2 l^2 C_i^2 + 4Q_{2i}^2|U_0|^8 + 2l^2k^2P_i^2 + Q_{1i}^2|U_0|^4 + 4k^2l^2(P_i + C_i)^2 + \left(X_1^2 + Y_1^2\right)^{1/2}.$$

The quantity $X_2$ is a positive constant. Hence, we have

$$\begin{aligned} \beta - h_2 \leqslant \beta - \Big\{\tfrac{1}{2}\big[&-2l^2(P_r Q_{1r} + P_i Q_{1i})|U_0|^2 - 4l^2(P_r Q_{2r} + P_i Q_{2i})|U_0|^4 \\ &- l^4 P_r^2 - 4k^2 l^2\left(P_i^2 + C_i^2\right)\big]\Big\}^{1/2}. \end{aligned}$$

It should be noted that the inequality $\mathrm{Im}(\Omega_1) < 0$ is satisfied as soon as

$$\beta - \Big\{\tfrac{1}{2}\big[-2l^2(P_r Q_{1r} + P_i Q_{1i})|U_0|^2 - l^4 P_r^2 - 4k^2 l^2\left(P_i^2 + C_i^2\right) + r_0\big]\Big\}^{1/2} < 0 \tag{18}$$

with

$$r_0 = -4l^2(P_r Q_{2r} + P_i Q_{2i})|U_0|^4.$$

Because $\beta$ is positive, the arrangements of (18) lead to

$$(P_r Q_{1r} + P_i Q_{1i}) - r < -\left(\frac{2\beta^2 + l^4 P_r^2 + 4k^2 l^2\left(P_i^2 + C_i^2\right)}{2l^2|U_0|^2}\right) < 0 \tag{19}$$

and necessarily

$$(P_r Q_{1r} + P_i Q_{1i}) - r < 0 \tag{20}$$

with $r$ defined by

$$r = \frac{r_0}{2l^2|U_0|^2} = -2(P_r Q_{2r} + P_i Q_{2i})|U_0|^2. \tag{21}$$

Expression (20) is the modulational instability criterion for Stokes waves in physical systems in which the MQCGLE holds. Moreover, this relation fulfils the well-known Lange and Newell's criterion [23]. Dealing with the work of these authors, it should be recalled that sinusoidal waves are subcritical (i.e., stable) for $P_r Q_{1r} + P_i Q_{1i} > 0$ and are supercritical (i.e., unstable) for $P_r Q_{1r} + P_i Q_{1i} < 0$.

To continue, we deduce from (20) that if $r > 0$, all supercritical waves are unstable under the modulation and subcritical waves with $r > (P_r Q_{1r} + P_i Q_{1i}) > 0$ are also unstable.

On the other hand, previous calculations are exploited to establish from the condition $\beta - h_2 > 0$ (i.e. $\mathrm{Im}(\Omega_1) > 0$)
that

$$(P_r Q_{1r} + P_i Q_{1i}) - r > 0. \tag{22}$$

This result means that Stokes waves which verify (22) are stable under modulation. We get from (22) that if $r < 0$, all subcritical waves are stable and supercritical waves with $r < (P_r Q_{1r} + P_i Q_{1i}) < 0$ are also stable under the modulation.

Let us examine the limiting case where the constraints $P_i = Q_{1i} = Q_{2i} = 0$ and $C = \gamma = 0$ act. Now, equation (2) becomes the cubic–quintic nonlinear Schrödinger equation [24] given by

$$\mathrm{i}U_t + P_r U_{xx} = Q_{1r}|U|^2 U + Q_{2r}|U|^4 U \tag{23}$$

and relations (19)–(21) lead to

$$P_r Q_{1r} - r_2 < -\left(\frac{l^2 P_r^2}{2|U_0|^2}\right) < 0$$

and necessarily

$$P_r Q_{1r} - r_2 < 0 \qquad \text{with} \qquad r_2 = -2P_r Q_{2r}|U_0|^2.$$

It can be mentioned explicitly that equation (23) governs the propagation of light beams in an inhomogeneous medium when the nonlinear polarization contains susceptibilities of third and fifth order [25], and also describes the boson gas with two- and three-body interactions [26]. This last result means that the present study also includes physical systems described by the cubic–quintic nonlinear Schrödinger equation and its derivative forms.

## 3. Exact solutions

The study made in the previous section presents the asymptotic behaviour of Stoke wave's solutions of equation (2). Indeed, it stresses the condition for which envelope solitons can be generated from the evolution of plane waves in the system. Knowing that systems described by equation (2) can support envelope solitons, we think it is good to fulfil out investigation by seeking some exact solutions of the MQCGLE. The presence of the quintic nonlinear term within this equation allows us to introduce a method based on the association of the Painlevé test theory and Hirota's bilinear technique [27, 28]. For this purpose, we start by rewriting equation (2) as

$$\mathrm{i}U_t + PU_{xx} = Q_1 U^2 V + Q_2 U^3 V^2 + C(U_x V_x / V) + \mathrm{i}\gamma U \tag{24a}$$

$$-\mathrm{i}V_t + P^* V_{xx} = Q_1^* V^2 U + Q_2^* V^3 U^2 + C^*(U_x V_x / U) - \mathrm{i}\gamma^* V \tag{24b}$$

where the asterisk denotes the complex conjugation. In the following analysis, $U$ and $V$ are assumed to be independent quantities. To obtain some exact solutions of equations (24), the modified Hirota's ansatz

$$U = \frac{G\,\mathrm{e}^{\mathrm{i}(Kx-\Omega t)}}{F^{\frac{1}{2}+\mathrm{i}\alpha}} \qquad V = \frac{H\,\mathrm{e}^{-\mathrm{i}(Kx-\Omega t)}}{F^{\frac{1}{2}-\mathrm{i}\alpha}} \tag{25}$$

are adopted; $K$, $\Omega$, $\alpha$ and $F(x,t)$ are considered as real. Expressions (25) are deduced from the truncation of the Puiseux expansions at the lowest level [29]. The report of relations (25) into equations (24) results in the set of equations below.

$$\begin{aligned}&\left(\frac{F}{Q_2 G}\right)\{\mathrm{i}D_{\alpha,t} + 2\mathrm{i}KPD_{\alpha,x} + PD^2_{\alpha,x} + \Omega - (P+C)K^2 - \lambda\}(GF) - \frac{\hat{\alpha}}{4}FF_{xx}\\ &\quad + \left(\frac{CF}{Q_2 GH}\right)\left\{2KF\,\mathrm{Im}(GH_x) + 2F_x\,\mathrm{Re}\left[\left(\frac{1}{2}+\mathrm{i}\alpha\right)GH_x\right] - FG_xH_x\right\}\\ &\quad - \left(\frac{1}{4}+\alpha^2\right)\frac{C}{Q_2}F_x^2 = \left\{\hat{\alpha}D^2_{0,x} + \frac{(\mathrm{i}\gamma-\lambda)}{Q_2} + \frac{Q_1}{Q_2}GH + G^2H^2\right\}(FF)\end{aligned} \tag{26a}$$

$$\begin{aligned}&\left(\frac{F}{Q_2^* H}\right)\{-\mathrm{i}D^*_{\alpha,t} - 2\mathrm{i}KP^*D^*_{\alpha,x} + P^*D^{*2}_{\alpha,x} + \Omega - (P^*+C^*)K^2 - \lambda^*\}(HF) - \frac{\hat{\alpha}^*}{4}FF_{xx}\\ &\quad + \left(\frac{C^*F}{Q_2^* GH}\right)\left\{2KF\,\mathrm{Im}(G_xH) + 2F_x\,\mathrm{Re}\left[\left(\frac{1}{2}-\mathrm{i}\alpha\right)G_xH\right] - FG_xH_x\right\}\\ &\quad - \left(\frac{1}{4}+\alpha^2\right)\frac{C^*}{Q_2^*}F_x^2 = \left\{\hat{\alpha}^*D^2_{0,x} + \frac{(-\mathrm{i}\gamma^*-\lambda^*)}{Q_2^*} + \frac{Q_1^*}{Q_2^*}GH + G^2H^2\right\}(FF).\end{aligned} \tag{26b}$$

In expressions (26), we have put $\hat{\alpha} = \left(\frac{1}{2}+\mathrm{i}\alpha\right)\left(\frac{3}{2}+\mathrm{i}\alpha\right)P/Q_2$ and the modified bilinear operator $D_{\alpha,x}$ is defined by

$$D_{\alpha,x}(GF) = \left\{\left[\frac{\partial}{\partial x} - \left(\frac{1}{2}+\mathrm{i}\alpha\right)\frac{\partial}{\partial x'}\right]G(x)F(\mathrm{x}')\right\}_{x=x'}.$$

Leaning on the idea introduced by Nozaki and Bekki [30], the complex constant $\lambda$ and $\alpha$ are calculated so that the right-hand sides of equations (26) become real. Hence, those parameters are subjected to the constraints $\alpha = \beta \pm (\beta^2 + 3/4)^{1/2}$; $Q_{2r}(\lambda_i - \gamma_r) = Q_{2i}(\lambda_r + \gamma_i)$ and $\beta = P_rQ_{2r} + P_iQ_{2i}/P_iQ_{2r} - P_rQ_{2i} = \mathrm{Re}(P/Q_2)/\mathrm{Im}(P/Q_2)$.

Equations (26) can be solved and its solutions may depend on the conditions applied on the parameters $K$, $\Omega$ as well as on the expansions chosen for the functions $F$, $G$ and $H$. Thus we will find possible bright soliton solutions of equations (26) by setting $K = \Omega = 0$ and considering the general forms [29]

$$U = \frac{a\,\mathrm{e}^{\theta}}{[1+\mathrm{e}^{\theta+\theta^*}]^{\frac{1}{2}+\mathrm{i}\alpha}} \qquad V = \frac{b\,\mathrm{e}^{\theta^*}}{[1+\mathrm{e}^{\theta+\theta^*}]^{\frac{1}{2}-\mathrm{i}\alpha}} \tag{27}$$

with $\theta = kx - \omega t$ where $k$ and $\omega$ are complex constants. By substituting (27) into equations (24) and solving the resultant set of equations for the coefficients $a$, $b$, $k$ and $\omega$, we obtain

$$\begin{aligned}&ab = 2k_r\sqrt{Z} \qquad ab > 0\\ &k_i = \frac{1}{A_0}[2(1+4\alpha^2)P_r - (C_r + 2\alpha C_i)]k_r + \frac{1}{A_0}(Q_{1r} + 2\alpha Q_{1i})\sqrt{Z} = ek_r + d\\ &\omega_r = [(1-e^2)P_i + 2eP_r - (1+e^2)C_i]k_r^2 + 2d[P_r - e(P_i + C_i)]k_r - [\gamma_r + (P_i + C_i)d^2]\\ &\omega_i = [(e^2-1)P_r + 2eP_i + (e^2+1)C_r]k_r^2 + 2d[P_i + e(P_r + C_r)]k_r - [\gamma_i - (P_r + C_r)d^2]\\ &k_r = A_1 \pm \sqrt{A_1^2 + B_1}\end{aligned} \tag{28}$$

with

$$A_0 = (1 + 4\alpha^2)P_i + 2\alpha(C_r + 2\alpha C_i) \qquad A_1 = \lfloor 2d(2\alpha - e)(P_i + C_i) - Q_{1i}\sqrt{Z}\rfloor / A_2$$
$$B_1 = [-(P_i + C_i)d^2 - \gamma_r]/A_2 \qquad A_2 = (e^2 - 4\alpha e)(P_i + C_i) + 8\alpha P_r + 3P_i - C_i$$

and

$$Z = \frac{1}{|Q_2|^2}\left[\left(\frac{3}{4} - \alpha^2\right)(P_r Q_{2r} + P_i Q_{2i}) - 2\alpha(P_i Q_{2r} - P_r Q_{2i}) - \left(\frac{1}{4} + \alpha^2\right)(C_r Q_{2r} + C_i Q_{2i})\right] \tag{29}$$

wherein the subscripts $r$ and $i$ denote real and imaginary quantities. From relation (28), it appears that $Z$ should always be taken positive so that $ab > 0$ by making an appropriate choice of the branch of $\alpha$. This bright soliton solution arises from a suitable compensation between the nonlinear and dispersive effects in the system. Moreover, solution (27) can be used to ensure the wave propagation in all systems from which equation (2) is derived. Unfortunately, we cannot get two or more envelope solitons since Hirota's method includes the asymmetric bilinear operators $D_{\alpha,x}, \ldots$ [29, 30].

When $K, \Omega \neq 0$, we look for shock-type wave solutions of equations (26) by considering the general forms [29]

$$U = \frac{a\,\mathrm{e}^{\mathrm{i}(Kx - \Omega t)}}{[1 + \mathrm{e}^{-2\mu(x - \eta t)}]^{\frac{1}{2} + \mathrm{i}\alpha}} \qquad V = \frac{b\,\mathrm{e}^{-\mathrm{i}(Kx - \Omega t)}}{[1 + \mathrm{e}^{-2\mu(x - \eta t)}]^{\frac{1}{2} - \mathrm{i}\alpha}}. \tag{30}$$

By direct substitutions we obtain all the unknown parameters of (30). Indeed, ansatz (30) satisfies equations (24) if

$$\begin{aligned}
& ab = 2\mu\sqrt{Z_1} \qquad ab > 0 \\
& K = -\frac{1}{A_0}[(1 + 4\alpha^2)P_r + 4(Q_{2r} + 2\alpha Q_{2i})Z_1]\mu - \frac{1}{A_0}(Q_{1r} + 2\alpha Q_{1i}))\sqrt{Z_1} = \sigma\mu + \delta \\
& \eta = 4\left[\left(\alpha + \frac{\sigma}{2}\right)P_r - \left(\alpha^2 + \alpha\sigma - \frac{1}{4}\right)P_i - Q_{2i}Z_1 + \left(\alpha^2 + \alpha\sigma + \frac{1}{4}\right)C_i\right]\mu \\
& \qquad + 2\delta(P_r - 2\alpha P_i) - 2Q_{1i}\sqrt{Z_1} - 4\alpha C_i\delta \\
& \Omega = [(P_r + C_r)\sigma^2 + 4Q_{2r}Z_1]\mu^2 + 2\lfloor\delta\sigma(P_r + C_r) + Q_{1r}\sqrt{Z_1}\rfloor\mu + [\gamma_i + (P_r + C_r)\delta] \\
& \mu = A_3 \pm \sqrt{A_3^2 + B_2}
\end{aligned} \tag{31}$$

with

$$A_3 = \lfloor -(P_i + C_i)\sigma\delta - Q_{1i}\sqrt{Z_1}\rfloor / \lfloor (P_i + C_i)\sigma^2 + 4Q_{2i}\sqrt{Z_1}\rfloor$$
$$B_2 = [\gamma_r - (P_i + C_i)\delta^2] / \lfloor (P_i + C_i)\sigma^2 + 4Q_{2i}\sqrt{Z_1}\rfloor$$

and $Z_1 = -Z$ where $Z$ is defined by equation (29). By taking an appropriate branch of $\alpha$, we can make $Z$ negative so that $ab > 0$. This shock-type solution (30) is generally a consequence of an overall balance among nonlinear and dispersive effects in the system when the former are greater than the latter. Moreover, this special solution can be utilized to describe pattern formation and/or study spatiotemporal transitions from coherent structures to chaotic states in physical systems in which the MQCGLE holds.

## 4. Conclusion

In this paper, a new wave equation (named the MQCGLE) has been formulated to describe dynamics of nonlinear waves in non-equilibrium systems. Examining the properties of this equation, we have not been able to derive any constant of motion attached to the MQCGLE although it is not certain that no more exist. This situation is quite understood since we know that the conventional CGLE, which is much simpler than equation (2), has no constant of motion.

Owing to a suite combination of the Painlevé analysis and Hirota's bilinear technique, some exact solutions (bright soliton and shock-type wave) of this new equation have been found. These solutions can be used to ensure the wave propagation, pattern formation and/or spatiotemporal transitions to chaos in the framework of the MQCGLE. Therefore, these solutions are of interest for a better understanding of the behaviour of the concerned nonlinear physical process and can thereby participate in the enhancement of the nonlinear fields.

Furthermore, it can be demonstrated that the known solutions of the limiting cases of the QCGLE can be recovered by taking the appropriate limit of the present solutions. In fact, when $C, \gamma = 0$, relations (27) and (30) lead respectively to the pulse and front solutions of the QCGLE obtained by Marcq *et al* [3].

On the other hand, we have exploited the Stoke wave analysis to built a result that gives the Lange and Newell's criterion for modulational instability (stability) of plane waves. We have obtained that this new criterion depends on the sign of the quantity $[(P_r Q_{1r} + P_i Q_{1i}) - r]$ in which $r$ represents the corrective term. For the sake of comparison, the modulational instability criterion (20) established in this work is quite similar to that obtained by Descalzi *et al* during their study of thermodynamic potentials for non-equilibrium systems [31]. Furthermore, result (20) is interesting because the present approach to its investigation is very different from the method based on the Lyapunov functional developed by Descalzi *et al* [31] and also from the method of cumulative momentum used by Lange and Newell [23].

Finally, since the existence of thermodynamic potentials is of fundamental importance in the macroscopic description of some physical systems, equation (2) can be exploited as a basic model to determine the thermodynamic potential for some class of non-equilibrium systems such as Descalzi *et al* [31] did, and the contribution of the term of factor $C$ will surely appear as a correction of the known results. It will also be of immense interest to appreciate the contribution of this term in the nonlinear Kuramoto–Sivashinsky phase equation which describes the weak turbulence dynamics of the extremum of the set of Lyapunov functionals associated with the MQCGLE.

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